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Generalization of Brittingham's localized solutions to the wave equation

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Abstract. A family of localized solutions of Brittingham's type is constructed for different cylindric coordinates. We use method of incomplete separation of variables with zero separation constant and, then, the Bateman transformation, which enables us to obtain solutions in the form of relatively undistorted progressing waves containing two arbitrary functions, each of which depends on a specific phase function.

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The purpose of this Rapid Note is generalization of the family of Brittingham's localized solutions to the wave equation known as the focus wave modes. In fact, they are presently represented by three specific solutions: Gauss [1], Hermite-Gauss [2], and Bessel-Gauss [3] modes. Most of reported Brittingham's type solutions [4–6] reduces to

$$\Psi\left(\rho,\varphi,z,\tau\right)=\frac{\rho^{m}\mathrm{e}^{\mathrm{i}m\varphi}}{\left(z-\tau\right)^{m+1}}v\left(z+\tau+\frac{\rho^{2}}{z-\tau}\right),\qquad(1)$$

where m is an integer, ρ,φ,z are the circular-cylinder coordinates, $\tau=ct$ is the time variable, c is the wavefront velocity, and v is an arbitrary function. Our investigation is connected with generalization of solutions of this particular type. In point of Courant and Hilbert's terminology [7], localized solutions (1) represent relatively undistorted progressing waves

$$\Psi\left(\mathbf{r},\tau\right)=g\left(\mathbf{r},\tau\right)f\left(\Phi\left(\mathbf{r},\tau\right)\right)\qquad(2)$$

where $\mathbf{r}$ defines an observation point in some coordinate system, $f\left(\Phi\right)$ is an arbitrary function with continuous partial derivatives while Φ and g are fixed functions, called the phase function and the distortion (or attenuation) factor. For Ψ being a solution of the wave equation, the phase function must satisfy the Hamilton-Jacobi equation

$$\left(\nabla\Phi\right)^{2}-\left(\partial\Phi/\partial\tau\right)^{2}=0.\qquad(3)$$

The undistorted progressing waves are of great importance for telecommunications, launching directional scalar and electromagnetic waves (missiles), and other applications.

Following [8], we construct the explicit solutions of the homogeneous wave equation

$$\left[\frac{1}{h_1h_2}\left(\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial}{\partial x_2}\right)\right)+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial\tau^2}\right]\Psi=0\qquad(4)$$

(h_1 and h_2 are the metric coefficients) in different orthogonal cylindric coordinate systems x_1,x_2,z in the form of a family of wavefunctions with invariable profiles $\Psi=w\left(x_1,x_2\right)v\left(\tau,z\right)$, that corresponds to incomplete separation of variables [9]. Putting the separation constant equal to zero we get the solution of the wave equation (4) as

$$\Psi=w\left(x_1\pm\mathrm{i}x_2\right)v\left(\tau\pm z\right)\qquad(5)$$

where w and v are arbitrary differentiable functions. Here v satisfies the 1D wave equation $\left(\partial^2/\partial z^2-\partial^2/\partial\tau^2\right)v=0$ while $w\left(x_1\pm\mathrm{i}x_2\right)$ meets $\left(\partial^2/\partial x_1^2+\partial^2/\partial x_2^2\right)w=0$, which for the rectangular (Cartesian), elliptic-cylinder, parabolic-cylinder, and bipolar coordinate systems is sufficient for satisfying the wave equation because $h_1=h_2$. For the remaining circular-cylinder coordinates, $x_1=\rho,\ x_2=\varphi$, equation (4) leads to another representation, $w\left(\rho,\varphi\right)=w\left(\rho\mathrm{e}^{\pm\mathrm{i}\varphi}\right)$, which is completely equivalent to Cartesian-coordinate form $w\left(x\pm\mathrm{i}y\right)$.

Finally, the whole wavefunction is subjected to one of the Bateman transformations [10] whose

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Cartesian-coordinate representation is

$$\Psi(x_0, y_0, z_0, \tau_0) \to \tilde{\Psi}(x_0, y_0, z_0, \tau_0) =$$
$$\frac{1}{z_0-\tau_0}\Psi\left(\frac{x_0}{z_0-\tau_0}, \frac{y_0}{z_0-\tau_0}, \frac{r_0^2-\tau_0^2-1}{2(z_0-\tau_0)}, \frac{r_0^2-\tau_0^2+1}{2(z_0-\tau_0)}\right),$$
$$x_0 = x/\lambda, \quad y_0 = y/\lambda, \quad z_0 = z/\lambda, \quad \tau_0 = \tau/\lambda,$$
$$r_0 = \sqrt{x_0^2+y_0^2+z_0^2}, \tag{6}$$

λ is a real constant parameter. The function $\tilde{\Psi}$ due to transformation (6) is as well a solution of the wave equation, and here we use the method applied in [5,12] for construction of new solutions from known wavefunctions. Expressing the transversal coordinates through their dimensionless Cartesian counterparts $x_1 = x_1(x_0, y_0)$, $x_2 = x_2(x_0, y_0)$ and fixing the phase sign, $v = v(\tau+z)$, one gets the general result for new wavefunctions

$$\tilde{\Psi} = \frac{1}{z_0-\tau_0} w(\phi)\, v(\Phi) \tag{7}$$

where ϕ is the transformed argument $x_1\left(\frac{x_0}{z_0-\tau_0}, \frac{y_0}{z_0-\tau_0}\right) \pm \mathrm{i}x_2\left(\frac{x_0}{z_0-\tau_0}, \frac{y_0}{z_0-\tau_0}\right)$ while Φ takes the form $z_0+\tau_0+\frac{x_0^2+y_0^2}{z_0-\tau_0}$, which is easily seen to be of type (2) with $g = \frac{w(\phi)}{z_0-\tau_0}$ and $f(\Phi) = v(\Phi)$. Thus, in the case of incomplete separation of variables with zero separation constant, g is defined within a factor $w(\phi)$, an arbitrary function of one prescribed argument.

Examining different cylindric coordinate systems, we get the following new families of localized wave structures:

(i) Rectangular coordinates, $x_1 = x,\ x_2 = y,\ h_1 = h_2 = 1$,

$$\tilde{\Psi} = \frac{\lambda}{z-\tau} w(\phi_\mathrm{r})\, v(\Phi_\mathrm{r}), \quad \phi_\mathrm{r} = \lambda\frac{x \pm \mathrm{i}y}{z-\tau}, \quad \Phi_\mathrm{r} = \frac{1}{\lambda}\left(z+\tau+\frac{x^2+y^2}{z-\tau}\right). \tag{8}$$

In the circular-cylinder coordinates, $x_1 = \rho,\ x_2 = \varphi$, $h_1 = 1,\ h_2 = \rho,\ -\infty < \rho < \infty,\ 0 \le \varphi < 2\pi,\ x = \rho\cos\varphi,\ y = \rho\sin\varphi$, one has equivalent representation written as

$$\tilde{\Psi} = \frac{\lambda}{z-\tau} w(\phi_\mathrm{c})\, v(\Phi_\mathrm{c}), \quad \phi_\mathrm{c} = \lambda\frac{\rho \mathrm{e}^{\pm \mathrm{i}\varphi}}{z-\tau}, \quad \Phi_\mathrm{c} = \frac{1}{\lambda}\left(z+\tau+\frac{\rho^2}{z-\tau}\right). \tag{9}$$

Localized waves (1) are easily seen to be a special case of (9) for $\lambda = 1$ and $w\left(\rho \mathrm{e}^{\mathrm{i}\varphi}\right) = \rho^m \mathrm{e}^{\mathrm{i}m\varphi}$.

(ii) Elliptic-cylinder coordinates, $x_1 = \mu,\ x_2 = \theta,\ h_1 = h_2 = a^2\left(\sinh^2\mu + \cos^2\theta\right),\ -\infty < \mu < \infty,\ 0 \le \theta < 2\pi,\ x = a\cosh\mu\cos\theta,\ y = a\sinh\mu\sin\theta,\ 0 < a < \infty$

$$\tilde{\Psi} = \frac{\lambda}{z-\tau} w(\phi_\mathrm{e})\, v(\Phi_\mathrm{e}), \quad \phi_\mathrm{e} = \frac{a}{z-\tau}\cosh(\mu \pm \mathrm{i}\theta), \quad \Phi_\mathrm{e} = \frac{1}{\lambda}\left(z+\tau+\frac{a^2}{z-\tau}\left(\sinh^2\mu+\cos^2\theta\right)\right). \tag{10}$$

(iii) Parabolic-cylinder coordinates, $x_1 = \zeta,\ x_2 = \chi,\ h_1 = h_2 = \zeta^2+\chi^2,\ 0 \le \zeta < \infty,\ -\infty < \eta < \infty,\ x = \zeta\chi,\ y = \frac{1}{2}\left(\zeta^2+\chi^2\right)$,

$$\tilde{\Psi} = \frac{\lambda}{z-\tau} w(\phi_\mathrm{p})\, v(\Phi_\mathrm{p}), \quad \phi_\mathrm{p} = \left(\frac{\lambda}{z-\tau}\right)^{\frac{1}{2}}(\zeta \pm \mathrm{i}\chi), \quad \Phi_\mathrm{p} = \frac{1}{\lambda}\left(z+\tau+\frac{\left(\zeta^2+\chi^2\right)^2}{4(z-\tau)}\right). \tag{11}$$

(iv) Bipolar coordinates, $x_1 = \xi,\ x_2 = \eta,\ h_1 = h_2 = a^2/\left(\cosh\xi - \cos\eta\right)^2,\ -\infty \le \xi < \infty,\ 0 \le \eta < 2\pi,\ x = a\sinh\xi/\left(\cosh\xi-\cos\eta\right),\ y = a\sin\eta/\left(\cosh\xi-\cos\eta\right),\ 0 < a < \infty$,

$$\tilde{\Psi} = \frac{\lambda}{z-\tau} w(\phi_\mathrm{b})\, v(\Phi_\mathrm{b}), \quad \phi_\mathrm{b} = \frac{a}{z-\tau}\operatorname{arccoth}\left(\frac{\xi \mp \mathrm{i}\eta}{2}\right), \quad \Phi_\mathrm{b} = \frac{1}{\lambda}\left(z+\tau+\frac{a^2}{z-\tau}\frac{\cosh\xi+\cos\eta}{\cosh\xi-\cos\eta}\right). \tag{12}$$

Note that ϕ's, as well as Φ's, satisfy the Hamilton-Jacobi equation (3), so all of them can be treated as the phase functions. Moreover, it can be verified that more general representation $\tilde{\Psi} = \frac{\lambda}{z-\tau}F(w(\phi), v(\Phi))$ is also possible, where F is as well an arbitrary differentiable function with respect to both arguments. Actually, the same is yielded if we replace the starting wavefunction (5) by $\Psi = F(x \pm \mathrm{i}y, \tau+z)$ or its equivalent for other coordinates. After transform (6) this leads to solution $\tilde{\Psi} = \frac{\lambda}{z-\tau}F(\phi, \Phi)$, equivalent to the previous result due to the arbitrariness of F.

On the basis of the curvilinear-coordinate representations (8–12) one can construct the field components for the case of TM electromagnetic waves as follows

$$E_1 = \frac{1}{h_1}\frac{\partial^2 u}{\partial x_1 \partial z},\ E_2 = \frac{1}{h_2}\frac{\partial^2 u}{\partial x_2 \partial z}, \quad E_z = \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial \tau^2},$$
$$B_1 = \frac{1}{h_2}\frac{\partial^2 u}{\partial x_2 \partial \tau},\ B_2 = -\frac{1}{h_1}\frac{\partial^2 u}{\partial x_1 \partial \tau},\ B_z = 0, \tag{13}$$

where the scalar function u is the Whittaker-Bromwich potential [11]. Notably, for the case of cylindric coordinate systems, the derivative $\partial u/\partial\tau$ is a solution of the wave equation (4). Therefore, putting $\partial u/\partial\tau = \tilde{\Psi}$, we get representation of the transversal components of the magnetic field in concrete cylindric coordinate systems:

Rapid Note

(i.a) Rectangular coordinates

$$B_x = \frac{1}{(z-\tau)^2}\left[2yw(\phi_{\mathrm{r}})\,v'(\Phi_{\mathrm{r}}) \pm \mathrm{i}\lambda^2 w'(\phi_{\mathrm{r}})\,v(\Phi_{\mathrm{r}})\right],$$
$$B_y = -\frac{1}{(z-\tau)^2}\left[2xw(\phi_{\mathrm{r}})\,v'(\Phi_{\mathrm{r}}) + \lambda^2 w'(\phi_{\mathrm{r}})\,v(\Phi_{\mathrm{r}})\right]; \tag{14}$$

(i.b) Circular-cylinder coordinates

$$B_\rho = \pm\mathrm{i}\frac{\lambda^2}{(z-\tau)^2}\mathrm{e}^{\pm\mathrm{i}\varphi} w'(\phi_{\mathrm{c}})\,v(\Phi_{\mathrm{c}}),$$
$$B_\varphi = -\frac{1}{(z-\tau)^2}\left[2\rho w(\phi_{\mathrm{c}})\,v'(\Phi_{\mathrm{c}}) + \lambda^2\mathrm{e}^{\pm\mathrm{i}\varphi} w'(\phi_{\mathrm{c}})\,v(\Phi_{\mathrm{c}})\right]; \tag{15}$$

(ii) Elliptic-cylinder coordinates

$$B_\mu = -\frac{1}{(z-\tau)^2}\frac{1}{\sinh^2\mu+\cos^2\theta}\Big[\sin(2\theta)\,w(\phi_{\mathrm{e}})\,v'(\Phi_{\mathrm{e}}) \mp \mathrm{i}\frac{\lambda}{a}\sinh(\mu\pm\mathrm{i}\theta)\,w'(\phi_{\mathrm{e}})\,v(\phi_{\mathrm{e}})\Big],$$
$$B_\theta = -\frac{1}{(z-\tau)^2}\frac{1}{\sinh^2\mu+\cos^2\theta}\Big[\sinh(2\mu)\,w(\phi_{\mathrm{e}})\,v'(\phi_{\mathrm{e}}) + \frac{\lambda}{a}\sinh(\mu\pm\mathrm{i}\theta)\,w'(\phi_{\mathrm{e}})\,v(\phi_{\mathrm{e}})\Big]; \tag{16}$$

(iii) Parabolic-cylinder coordinates

$$B_\zeta = \frac{1}{(z-\tau)^2}\left[\chi w(\phi_{\mathrm{p}})\,v'(\Phi_{\mathrm{p}}) \pm \mathrm{i}\frac{\left(\lambda^3(z-\tau)\right)^{\frac{1}{2}}}{\zeta^2+\chi^2}w'(\phi_{\mathrm{p}})\,v(\Phi_{\mathrm{p}})\right],$$
$$B_\chi = -\frac{1}{(z-\tau)^2}\left[\zeta w(\phi_{\mathrm{p}})\,v'(\Phi_{\mathrm{p}}) + \frac{\left(\lambda^3(z-\tau)\right)^{\frac{1}{2}}}{\zeta^2+\chi^2}w'(\phi_{\mathrm{p}})\,v(\Phi_{\mathrm{p}})\right]; \tag{17}$$

(iv) Bipolar coordinates

$$B_\xi = -\frac{2}{(z-\tau)^2}\left[\cosh\xi\sin\eta\; w(\phi_{\mathrm{b}})\,v'(\Phi_{\mathrm{b}}) \pm \mathrm{i}\frac{\lambda}{a}\frac{(\cosh\xi-\cos\eta)^2}{4-(\xi\mp\mathrm{i}\eta)^2}w'(\phi_{\mathrm{b}})\,v(\Phi_{\mathrm{b}})\right],$$
$$B_\eta = \frac{2}{(z-\tau)^2}\left[\sinh\xi\cos\eta\; w(\phi_{\mathrm{b}})\,v'(\Phi_{\mathrm{b}}) - \frac{\lambda}{a}\frac{(\cosh\xi-\cos\eta)^2}{4-(\xi\mp\mathrm{i}\eta)^2}w'(\phi_{\mathrm{b}})\,v(\Phi_{\mathrm{b}})\right]. \tag{18}$$

We are reminded that in all above formulas w and v are arbitrary differentiable functions, w' and v' denote their derivatives. As it is easily seen, the localization properties of the above scalar and electromagnetic waves can provided by corresponding choice of the functions w and v. Well-known replacement $(\mathbf{E}_{\mathrm{TM}}, \mathbf{B}_{\mathrm{TM}}) \to (\mathbf{B}_{\mathrm{TE}}, -\mathbf{E}_{\mathrm{TE}})$ allows using the same expressions for description of the non-zero electric field strength components in case of electromagnetic waves of TE type. In the general case, obtaining $\mathbf{E}_{\mathrm{TM}}$ and $\mathbf{B}_{\mathrm{TE}}$ vectors involves integration with respect to τ, which requires special consideration, in particular, specifying corresponding conditions. However, a subset of possible solutions for all six components of the electromagnetic field can be obtained if we consider the scalar wavefunction as the potential itself rather than as its time derivative, that is, if we put $u = \frac{\lambda}{z-\tau}w(\phi)\,v(\Phi)$ or even $u = \frac{\lambda}{z-\tau}F(\phi,\Phi)$. One can check by direct calculations that corresponding components of the electromagnetic field constructed with relations (13) satisfy the homogeneous Maxwell equation.

As the phase function ϕ is essentially complex, concrete real-valued solutions are given by real or imaginary part of the complex relationships. For example, choosing the rectangular coordinates and $w = (x_0 + \mathrm{i}y_0)^2$ we get two real-valued solutions $\tilde{\Psi}_{\mathrm{Re}} = \lambda(z-\tau)^{-3}\left(x^2-y^2\right)v(\Phi_{\mathrm{r}})$ and $\tilde{\Psi}_{\mathrm{Im}} = 2\lambda(z-\tau)^{-3}\,xyv(\Phi_{\mathrm{r}})$. Separation of the real and imaginary parts requires individual consideration for each particular type of the function w.

Note that applying to obtained waves (8–12) and (14–18) the linear transform $\tau \to (\tau+\beta z)/\epsilon$, $z \to (z+\beta\tau)/\epsilon$, $\epsilon = \sqrt{1-\beta^2}$, β is an arbitrary complex parameter, which is also invariant with respect to the wave operator, one can demonstrate that they can be rewritten in more general form, in which real factors λ and $1/\lambda$ are replaced by two independent complex parameters.

All previous consideration was constrained to solutions with the phase function Φ of type $z+\tau+\rho^2/(z-\tau)$. Remarkably, additional application of the transverse transform $\tau \to (\tau+\beta x)/\epsilon$, $x \to (x+\beta\tau)/\epsilon$ leads as well to a solution of the wave equation, but with the phase function of another, non-axisymmetric type

$$\tilde{\Psi}_\perp = \frac{\epsilon\lambda}{\epsilon z-\beta x-\tau}w\left(\lambda\frac{x\pm\mathrm{i}\epsilon y+\beta\tau}{\epsilon z-\beta x-\tau}\right) \times v\left(\frac{1}{\epsilon\lambda}\left(z+\beta x+\tau+\frac{(x+\beta\tau)^2+\epsilon^2y^2}{\epsilon z-\beta x-\tau}\right)\right). \tag{19}$$

Other examples of phase functions, promising for constructing relatively undistorted progressing waves, are $z\pm\tau+(\rho+\lambda)^2/(z\mp\tau)$ and $\rho\pm\tau+z^2/(\rho\mp\tau)$. The last phase function, as well as Ψ defined by (8), is a particular case of more general relation that in the spheric coordinates r,ϑ,φ takes the form

$$\Phi_{\mathrm{s}} = \frac{\tau^2-r^2}{\tau-r\cos\Theta},\qquad \cos\Theta = \cos\vartheta\cos\vartheta_0+\sin\vartheta\sin\vartheta_0\cos(\varphi-\varphi_0), \tag{20}$$

where ϑ_0 and φ_0 are constants [13]. Investigation and possible generalization of such waves is one among future extensions of this work.

Although the primary goal of this investigation is generalization of localized waves, the initial expressions that were used for their construction, namely, a family of non-axisymmetric invariable-profile wavefunctions (5), are of independent interest. They present another direction of development of this research in context of widely discussing investigations on beams with invariant transverse structure. Choosing different complex transverse factors $w\,(x_1 \pm \mathrm{i}x_2)$ and separating the real or imaginary part, one can obtain beams of desired structure, which remains unchanged in both transverse and longitudinal directions. Wavefunctions (5) can be treated as generalization of plane waves, and traditional approach allows even solving problems of reflection and refraction of scalar and electromagnetic waves at the plane boundaries. In the case of electromagnetic waves all $\mathbf{E}$ and $\mathbf{B}$ components can be constructed using the scalar wavefunction as the Whittaker-Bromwich potential.

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